

NODAL SOLUTIONS FOR LANE-EMDEN PROBLEMS IN ALMOST-ANNULAR DOMAINS

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ABSTRACT. In this paper we prove an existence result to the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N which is a perturbation of the annulus. Then there exists a sequence $p_1 < p_2 < \dots$ with $\lim_{k \rightarrow +\infty} p_k = +\infty$ such that for any real number $p > 1$ and $p \neq p_k$ there exist at least one solution with m nodal zones. In doing so, we also investigate the radial nodal solution in an annulus: we provide an estimate of its Morse index and analyze the asymptotic behavior as $p \rightarrow 1$.

Keywords: semilinear elliptic equations, nodal solutions, supercritical problems.

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1. INTRODUCTION

We are interested in the existence of nodal solutions to the Lane-Emden problem

$$(1.1) \quad \begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$ and Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 2$.

Addressing this problem in full generality is hard, and the answer changes according to the features (geometrical or topological) of the domain Ω and on the exponent p of the nonlinear term. A wide literature is available on this subject, and many interesting results have been obtained. For example, if $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and for any p when $N = 2$, the compactness of the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$ gives the existence of *infinitely many* solutions, to (1.1) in any smooth domain Ω . On the other hand, when the exponent p becomes critical or supercritical, i.e. $p \geq \frac{N+2}{N-2}$ for $N \geq 3$, the compactness of the previous embedding can fail and so does in general the existence of solutions. Indeed the classical Pohozaev identity [Po] implies that in this case, if Ω is starshaped with respect to one of its point, then (1.1) does not admit solutions. The existence can be restored when the domain Ω exhibits an hole. The simplest example is the case of the annulus where a radial solution always exists even if the exponent p is supercritical. We quote also the papers [BC] and [Cor] where the existence of a positive solution is proved in the critical case in a general domain with holes. If $p > \frac{N+2}{N-2}$ the existence of positive solutions have been established in [dPW] in domains with a small circular hole, while [DW] examines the case of nodal solutions. Both these papers rely on a perturbation argument

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around the exterior domain. Finally, we want to quote the paper [BCGP] where the existence of a positive solution in an expanding annular type domain is proved if the radius of the domain is large enough (see also the references therein for other existence results). Here we focus onto domains Ω which are small perturbations of an annulus, namely

$$(1.2) \quad \begin{aligned} \Omega_t &= \{x + t\sigma(x) : x \in A\}, \\ \text{where } A &= \{x \in \mathbb{R}^N : a < |x| < b\} \text{ is an annulus} \\ \text{and } \sigma : \bar{A} &\rightarrow \mathbb{R}^N \text{ is a smooth function.} \end{aligned}$$

This perturbation has been used in [DD] to study the Gelfand problem on a deformed ball, and later also by [Cow] for the Hénon problem.

Here we prove existence of positive or nodal solutions to (1.1) if the exponent of the nonlinear term is different from a sequence of values that accumulates at $+\infty$. Our result does not depend on the measure of the annulus A that can be small or large and does not depend on the shape of the hole and produces a nodal solution whose profile is close to the radial nodal solution in the annulus. Some existence results in domains which are very general perturbations of a fixed domain Ω have been obtained in [D] using the Leray-Schauder degree in the subcritical case. Finally let us observe that it is one of the very few results for nodal solutions in the supercritical range.

Our main result is the following:

Theorem 1.1. *Let m be a positive integer and p a real number greater than 1. Then there exists a sequence of exponents $1 < p_1 < p_2 < \dots < p_k \nearrow +\infty$ such that for $p \neq p_k$ there exists a classical solution of (1.1) with m nodal regions in $\Omega = \Omega_t$ for t small enough.*

In the case of a large annulus, i.e. $a = R$ and $B = R + 1$ with R large enough Theorem 1.1 extends the existence result in [BCGP] to a more general annular type expanding domain and to the case of nodal solutions. Let us stress that our proof looks like easier.

The previous result relies on the use of the Implicit Function Theorem once we have studied the linearized problem associated to (1.1) when Ω is an annulus and u is its radial solutions (see Section 2 for the properties of this solution). Our main result in this direction, which in our opinion is interesting itself, is given by the following proposition,

Proposition 1.2 (Characterization of degeneracy). *Let Ω be an annulus of \mathbb{R}^N with $N \geq 2$ and v_p a radial solution to (1.1) with m nodal zones. Then v_p is radially nondegenerate, and it is degenerate if and only if*

$$(1.3) \quad \nu_l(p) = -j(N-2+j), \quad \text{for some } l = 1, \dots, m \text{ and } j \geq 2$$

or

$$(1.4) \quad \nu_m(p) = -(N-1)$$

where $\nu_l(p)$ is the l^{th} eigenvalue of problem (3.3).

In some sense the previous proposition characterizes the “bad values” of p where the Implicit Function Theorem does not hold and generalizes the study of the degeneracy of the radial positive solution in the paper [GGPS] to the case of nodal solutions. Since it is not possible to solve the equations (1.3), (1.4) explicitly, we will

derive that they have a countable number of solutions by studying them for p close to 1 and $+\infty$ and applying some ideas of [DW]. This allows to prove the following:

Proposition 1.3 (Degeneracy points). *Let Ω be an annulus in \mathbb{R}^N with $N \geq 2$ and v_p a radial solution to (1.1) with m nodal zones. Then there exists a sequence $1 < p_1 < p_2 < \dots < p_k \nearrow +\infty$ such that v_p is degenerate if and only if $p = p_k$. Moreover the Morse index of v_p goes to $+\infty$ as $p \rightarrow \infty$.*

Propositions 1.2 and 1.3 extend some properties of the radial solution v_p studied in [PS] to the case of nodal solutions with $m \geq 2$. The characterization of degeneracy in Proposition 1.2 is the key ingredient in [GGPS] to prove the bifurcation of nonradial solutions from the positive radial solution in the annulus. Unfortunately in the case of nodal solutions some technical problems do not allow to conclude. We believe anyway that this problem deserves further study.

Another interesting byproduct of Proposition 1.2 is an estimate from below for the Morse index.

Proposition 1.4 (Morse index). *Let Ω be an annulus in \mathbb{R}^N with $N \geq 2$, $p > 1$ and v_p a radial solution to (1.1) with m nodal zones. Then its Morse index is strictly greater than $(m-1)(N+1)$.*

Such estimate improves the ones obtained in [AP, Theorem 1.1] and [BD, Theorem 2.2] in the particular case of power nonlinearity.

The paper is organized as follows: in Section 2 we recall some properties of the radial solution to (1.1) in the annulus, in Section 3 we study the degeneracy of the radial solution, we prove Proposition (1.2) and we study the set of solutions of the equation (1.3) obtaining Proposition 1.3 from a careful study of the asymptotic of the radial solution as $p \rightarrow 1$. Finally in Section 4 we prove Theorem (1.1) and some qualitative properties of the solution.

2. PRELIMINARIES ON RADIAL SOLUTIONS IN THE ANNULUS

Let $A = \{x \in \mathbb{R}^N : a < |x| < b\}$ be an annulus and $N \geq 2$. We focus here on radial solutions to the problem

$$(2.1) \quad \begin{cases} -\Delta v = |v|^{p-1}v & \text{in } A, \\ v = 0 & \text{on } \partial A, \end{cases}$$

which have precisely m nodal zones. Since v and $-v$ solve (2.1) we fix the sign of the solution assuming that $v'(a) > 0$.

For $m = 1$, we are actually looking at positive solutions to

$$(2.2) \quad \begin{cases} -\Delta u = u^p & \text{in } A, \\ u > 0 & \text{in } A, \\ u = 0 & \text{on } \partial A. \end{cases}$$

Problem (2.2) has an unique radial solution (see, for instance, [NN]), which we denote by u_p . It is radially nondegenerate for all p , and nondegenerate for all p except an increasing sequence $1 < p_1 < p_2 < \dots < p_k \nearrow +\infty$ (see [GGPS, Lemma 2.3 and Section 4] for details).

For $m \geq 2$, existence of a solution for (2.1) comes from a standard application of the Nehari method. For $a \leq \alpha < \beta \leq b$, we write $A(\alpha, \beta)$ for the annulus with

radii α and β and $H(\alpha, \beta)$ for $H_{0,r}^1(A(\alpha, \beta))$, the space of radial functions belonging to $H_0^1(A(\alpha, \beta))$. On every $H(\alpha, \beta)$, we may define the energy functional

$$\mathcal{E}(v) = \frac{1}{2} \int_{A(\alpha, \beta)} |\nabla v|^2 - \frac{1}{p+1} \int_{A(\alpha, \beta)} |v|^{p+1},$$

and the set

$$\mathcal{N}(\alpha, \beta) = \left\{ v \in H(\alpha, \beta) : \int_{A(\alpha, \beta)} |\nabla v|^2 = \int_{A(\alpha, \beta)} |v|^{p+1} \right\}.$$

It is well known that the nontrivial positive radial solution of the problem (2.1) in the annulus $A(\alpha, \beta)$ is a critical value of \mathcal{E} , that can be seen as a Mountain Pass point on $H(\alpha, \beta)$ or as a minimum point on $\mathcal{N}(\alpha, \beta)$. A nodal radial solution with exactly m nodal zones and zeros $a = r_0 < r_1 < r_2 < \dots < r_m = b$ can be produced by solving the minimization problem

$$(2.3) \quad \Lambda(r_1, \dots, r_{m-1}) = \min \left\{ \sum_{i=1}^m \inf_{\mathcal{N}(r_{i-1}, r_i)} \mathcal{E} : a = r_0 < r_1 < \dots < r_m = b \right\}.$$

Theorem 2.1 (Existence and uniqueness of the radial solution). *Let $p > 1$ and m be a positive integer. Problem (2.1) admits exactly one radially symmetric nodal solution $v_p = v_p(r)$ with precisely m nodal zones and $v_p'(a) > 0$. Moreover such solution realizes the minimum of (2.3).*

We do not report the details of the existence part of the proof, which are very next to [BW93, Theorem 2.1], and somehow easier (see also Remark 2.2.a in the same paper). We also mention [DW], where the same method is applied. Concerning uniqueness, it has been established in [NN, Theorem 3.1]

Remark 2.2. *Let v_p be the radial solution of (2.1) and $a = r_0 < r_1 < \dots < r_m = b$ its zeros. Then $u_i(x) = (-1)^{i-1} v_p(x) 1_{\{r_{i-1} \leq |x| \leq r_i\}} \in \mathcal{N}(r_{i-1}, r_i)$ is the only positive radial solution to (2.2) in the annulus $A(r_{i-1}, r_i)$, as $i = 1, \dots, m$. We recall for future convenience that every u_i is radially nondegenerate and its radial Morse index is 1.*

3. THE LINEARIZATION AT v_p .

In this section we investigate the nondegeneracy of v_p , precisely we want to characterize the values of p such that the linearized problem

$$(3.1) \quad \begin{cases} -\Delta w = p|v_p|^{p-1}w & \text{in } A, \\ w = 0 & \text{on } \partial A \end{cases}$$

has a nontrivial solution. As standard, we decompose any solution w along Y_k , the space of the eigenfunctions of the Laplace-Beltrami operator on the sphere \mathbb{S}^{N-1} , and write

$$w(x) = \sum_{k=0}^{\infty} \phi_k(r) Y_k(\theta), \quad a < r < b, \quad \theta \in \mathbb{S}^{N-1}.$$

The components ϕ_k then satisfy the differential equations

$$(3.2) \quad \begin{cases} -\phi_k'' - \frac{N-1}{r} \phi_k' = \left(p|v_p|^{p-1} - \frac{\lambda_k}{r^2} \right) \phi_k & a < r < b, \\ \phi_k(a) = \phi_k(b) = 0, \end{cases}$$

where λ_k is the eigenvalue associated to Y_k , i.e. $\lambda_k = j(N-2+j)$ for some $j \in \mathbb{N}$. We also address to the one-dimensional problem

$$(3.3) \quad \begin{cases} -\phi'' - \frac{N-1}{r}\phi' = \left(p|v_p|^{p-1} + \frac{\nu}{r^2}\right)\phi & a < r < b, \\ \phi(a) = \phi(b) = 0. \end{cases}$$

The Sturm-Liouville theory guarantees that all the eigenvalues of (3.3) are simple and that are characterized as min-max:

$$(3.4) \quad \nu_l(p) = \inf_{\dim(V)=l} \max_{\phi \in V} \frac{\int_a^b r^{N-1} (|\phi'|^2 - p|v_p|^{p-1}\phi^2) dr}{\int_a^b r^{N-3}\phi^2 dr},$$

where V runs through subspaces of $H_{0,r}^1(A)$.

Proof of Proposition 1.2. Comparing (3.3) and (3.2), it is clear that v_p is radially degenerate only if $\nu_l(p) = -\lambda_0 = 0$ for some l , and degenerate if there exist l and k such that $\nu_l(p) = -\lambda_k$.

By the min-max characterization (3.4), it is immediately seen that (3.3) has at least m negative eigenvalues, because the functions u_i introduced in Remark 2.2 have disjoint supports and they all satisfy

$$\begin{aligned} \int_a^b r^{N-1} (|u_i'|^2 - p|v_p|^{p-1}u_i^2) dr &= \int_{r_{i-1}}^{r_i} r^{N-1} (|u_i'|^2 - p|u_i|^{p+1}) dr \\ &= -(p-1) \int_{r_{i-1}}^{r_i} r^{N-1} |u_i'|^2 dr < 0. \end{aligned}$$

Next claim concerns the $(m+1)^{th}$ eigenvalue.

Claim: the $(m+1)^{th}$ eigenvalue of (3.3) is positive.

To show this we look at the auxiliary function $z = rv_p' + \frac{2}{p-1}v_p$, which satisfies the equation in (3.3) with $\nu = 0$, but not the boundary condition. Let us prove that z has exactly m zeroes. Actually, as v_p is the positive radial solution of (1.1) in the annulus $A(r_{i-1}, r_i)$ ($i = 1, \dots, m$), it follows that $z(r_{i-1}) = r_{i-1}v_p'(r_{i-1})$ and $z(r_i) = r_i v_p'(r_i)$ are nonzero (otherwise v_p and v_p' should vanish at the same point, implying $v_p \equiv 0$) and have opposite sign. Hence z has at least one zero in any sub-interval, and $z' \neq 0$ at any point where $z = 0$ (otherwise also $z \equiv 0$).

Finally z has not more than two nodal zones in any sub-interval (r_{i-1}, r_i) because otherwise it should be a sign-changing eigenfunction on a subdomain, contradicting the fact that v_p has radial Morse index one in the annulus $A(r_{i-1}, r_i)$.

On the other hand, the $(m+1)^{th}$ eigenfunction of (3.3) has $m+2$ zeroes in $[a, b]$, by the classical Sturm Liouville Theorem. If the $(m+1)^{th}$ eigenvalue $\nu_{m+1}(p)$ where nonpositive, we could apply the Sturm-Picone Comparison Theorem and obtain that z has at least $m+1$ zeros and this gives a contradiction proving the claim.

In particular, this shows that the Morse index of problem (3.3) is m and $\nu_l(p) \neq 0$ for every l . Then v_p is radially nondegenerate, and the equality $\nu_l(p) = -j(N-2+j)$ can hold only for $l \leq m$ and $j \geq 1$. Actually if $j = 1$, the equality $\nu_l(p) = -(N-1)$ can hold only for $l = m$, because $\nu_1(p) < \dots < \nu_{m-1} < -(N-1)$ for all p . To see this fact, we introduce another auxiliary function $\zeta := v_p'$: it solves

$$-\zeta'' - \frac{N-1}{r}\zeta' = \left(p|v_p|^{p-1} - \frac{N-1}{r^2}\right)\zeta$$

and it has at least m zeros inside (a, b) . Comparing this equation with (3.3) by means of Sturm-Picone Comparison principle yields that, if $-(N-1) \leq \nu_l(p)$, then the related eigenfunction should have at least $m-1$ internal zeros, i.e. $l \geq m$. \square

The characterization of Proposition 1.2 allows to compute the Morse index of radial solutions, even though in a not completely explicit way. Anyway it suffices to give an estimate from below. We prove here Proposition 1.4.

Proof of Proposition 1.4. As explained in [GGPS, Lemmas 2.1 and 2.2], the Morse index of the radial solution v_p is exactly the sum of the dimensions of the eigenspace of the spherical harmonics (related to $j(N-2+j)$) such that $\nu_l(p) + j(N-2+j) < 0$ for some $j \geq 1$ and for some $l = 1, \dots, m$, i.e.

$$(3.5) \quad m(v_p) = \sum_{l=1}^m \sum_{j < J_l(p)} \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!},$$

where $J_l(p) = \left(\sqrt{(N-2)^2 - 4\nu_l(p)} - N + 2 \right) / 2$. On the other hand in the proof of Proposition 1.2 it has been showed that $\nu_l(p) < -(N-1)$ for $l = 1, \dots, m-1$ and $\nu_m(p) < 0$. Hence $J_l(p) > 1$ for $l = 1, \dots, m-1$ and $J_m(p) > 0$, so that

$$m(v_p) \geq (m-1) \sum_{j=0}^1 \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!} + 1 = (m-1)(N+1) + 1.$$

\square

Next step stands in showing that the equality $\nu_l(p) = -j(N-2+j)$ is satisfied for a discrete increasing sequence of values of p_k . We shall deduce this fact by examining the behavior of the eigenvalues $\nu_l(p)$ when p approaches the ends of the existence range (i.e. $p \rightarrow +\infty$ and $p \rightarrow 1$) and then taking advantage of a sort of “local analiticity” of the map $p \mapsto \nu_l(p)$. The asymptotic behavior of v_p as $p \rightarrow +\infty$ has been deeply investigated in [PS]. To our purpose it suffices to check that all the negative eigenvalues diverge.

Lemma 3.1. *As $p \rightarrow +\infty$, it holds that $\nu_l(p) \rightarrow -\infty$ for $l = 1, \dots, m$.*

Proof. By the min-max characterization of eigenvalues (3.4), we have that

$$\nu_1 < \dots < \nu_m \leq \max \left\{ \frac{\int_a^b r^{N-1} (|\phi'|^2 - p|v_p|^{p-1}\phi^2) dr}{\int_a^b r^{N-3}\phi^2 dr} : \phi = \sum_{i=1}^m c_i u_i \right\}$$

where $u_i \in \mathcal{N}(r_{i-1}, r_i)$ are the positive solutions introduced in Remark 2.2. Now

$$\begin{aligned} \frac{\int_a^b r^{N-1} (|\phi'|^2 - p|v_p|^{p-1}\phi^2) dr}{\int_a^b r^{N-3}\phi^2 dr} &= \frac{\sum_{i=1}^m c_i^2 \int_{r_{i-1}}^{r_i} r^{N-1} (|u_i'|^2 - p u_i^{p+1}) dr}{\sum_{i=1}^m c_i^2 \int_{r_{i-1}}^{r_i} r^{N-3} u_i^2 dr} \\ &= (1-p) \frac{\sum_{i=1}^m c_i^2 \int_{r_{i-1}}^{r_i} r^{N-1} |u_i'|^2 dr}{\sum_{i=1}^m c_i^2 \int_{r_{i-1}}^{r_i} r^{N-3} u_i^2 dr} \leq (1-p) a^2 \lambda_1 \end{aligned}$$

where λ_1 denotes the first eigenvalue of the Laplacian with zero Dirichlet boundary conditions on A . So the claim follows. \square

Next we analyze the behavior of v_p for p close to 1. The following result is in the spirit of [G], where a detailed asymptotic picture is obtained in a more general framework, after assuming a-priori that $\|v_p\|_2^{p-1}$ is bounded. Here we are able to prove that actually $\|v_p\|_\infty^{p-1}$ stays bounded, and then deduce that a suitable rescaling of v_p converges to an eigenfunction of the Laplacian.

Proposition 3.2. *Let λ_m be the m^{th} radial eigenvalue for the Laplacian in A and ψ_m be the corresponding radial eigenfunction. Then*

$$(3.6) \quad \|v_p\|_\infty^{p-1} \rightarrow \lambda_m \quad \text{as } p \rightarrow 1,$$

and

$$(3.7) \quad \frac{v_p}{\|v_p\|_\infty} \rightarrow \psi_m \quad \text{in } C^2(A), \quad \text{as } p \rightarrow 1.$$

Proof. We first show that $\|v_p\|_\infty^{p-1}$ is bounded near $p = 1$. We assume by contradiction that there exists a sequence $p_n \rightarrow 1$ such that

$$t_n = \|v_{p_n}\|_\infty^{\frac{p_n-1}{2}} \rightarrow \infty \quad \text{as } n \rightarrow +\infty,$$

and take $q_n \in (a, b)$ a maximum point for $|v_{p_n}(r)|$. Up to an extracted sequence, q_n converges to some $q_0 \in [a, b]$. Let us show that

$$(3.8) \quad t_n(b - q_n) \not\rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

To see this let us denote by r_n the last internal zero of v_{p_n} , and notice that $b - q_n > (b - r_n)/2$: this is obvious if the maximum point q_n does not belong to the last nodal region, otherwise it follows by the Gidas, Ni, Nirenberg monotonicity property [GNN, Theorem 2]. So, in order to prove (3.8) it suffices to check that $t_n(b - r_n) \not\rightarrow 0$.

We thus look at the rescaled function $\tilde{v}_n(r) = \frac{1}{\|v_{p_n}\|} v_{p_n}(r_n + \frac{r-1}{t_n})$, which satisfies

$$\begin{cases} -\tilde{v}_n'' = \frac{N-1}{t_n r_n + r - 1} \tilde{v}_n' + \tilde{v}_n^{p_n}, & r \in I_n = (1, 1 + t_n(b - r_n)), \\ 0 < \tilde{v}_n(r) \leq 1, & r \in I_n \\ \tilde{v}_n(1) = 0 = \tilde{v}_n(1 + t_n(b - r_n)). \end{cases}$$

Multiplying the equation by \tilde{v}_n and integrating by parts gives

$$\begin{aligned} \int_{I_n} |\tilde{v}_n'|^2 dr &= \int_{I_n} \left(\frac{N-1}{t_n r_n + r - 1} \tilde{v}_n \tilde{v}_n' + \tilde{v}_n^{p_n+1} \right) dr \\ &\leq \frac{N-1}{t_n a} \left(\int_{I_n} |\tilde{v}_n'|^2 dr \right)^{\frac{1}{2}} \left(\int_{I_n} \tilde{v}_n^2 dr \right)^{\frac{1}{2}} + \int_{I_n} \tilde{v}_n^2 dr \\ &\leq \left(\frac{N-1}{a} (b - r_n) + (t_n(b - r_n))^2 \right) \int_{I_n} |\tilde{v}_n'|^2 dr \end{aligned}$$

by Poincaré inequality, which implies that (3.8) holds.

A similar argument will be used to show that

$$(3.9) \quad t_n(q_n - a) \not\rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

If by contradiction (3.9) does not hold, we must have that $q_n \rightarrow a$; then denoting by s_n the first internal zero of v_{p_n} and reasoning as before yields that $t_n(s_n - a)$ is bounded away from zero. If q_n is not contained in the first nodal region, then

$t_n(q_n - a)$ does not vanish. Otherwise, the same monotonicity argument applied to the Kelvin transform of v_{p_n} yields that

$$q_n - a > a \left(\left(\frac{1 + (s_n/a)^{2-N}}{2} \right)^{\frac{1}{2-N}} - 1 \right).$$

Since we are assuming that $q_n \rightarrow a$, it follows that also $s_n \rightarrow a$. So, for large values of n , the right-hand side behaves like $(s_n - a)/2$ and therefore we conclude that (3.9) holds.

Next we introduce the auxiliary function

$$u_n(r) = \frac{1}{\|v_{p_n}\|_\infty} v_{p_n} \left(q_n + \frac{r}{t_n} \right),$$

that satisfies

$$\begin{cases} -u_n'' - \frac{N-1}{t_n q_n + r} u_n' = |u_n|^{p_n-1} u_n, & \text{in } (\alpha_n, \beta_n), \\ |u_n(r)| \leq |u_n(0)| = 1, \quad u_n'(0) = 0, \\ u_n(\alpha_n) = 0 = u_n(\beta_n). \end{cases}$$

Here $\alpha_n = t_n(a - q_n)$ and $\beta_n = t_n(b - q_n)$. By (3.8) and (3.9), as n goes to infinity, the set (α_n, β_n) goes to an unbounded interval I containing 0. To fix notations, we take $I = (\alpha_o, +\infty)$ with $\alpha_o < 0$. Besides

$$\begin{aligned} |u_n'(r)| &= \left| \int_0^r u_n'' d\rho \right| = \left| \int_0^r \left(\frac{N-1}{t_n q_n + \rho} u_n' + |u_n|^{p_n-1} u_n \right) d\rho \right| \\ &\leq \int_0^r \frac{N-1}{t_n q_n + \rho} |u_n'| d\rho + r \end{aligned}$$

if $r > 0$, or

$$|u_n'(r)| \leq \int_r^0 \frac{N-1}{t_n q_n + \rho} |u_n'| d\rho - r$$

if $r < 0$. So by Gronwall's Lemma we deduce that $|u_n'(r)| \leq r \left(\frac{t_n q_n + r}{t_n q_n} \right)^{N-1}$ if $r > 0$,

or $|u_n'(r)| \leq |r| \left(\frac{t_n q_n}{t_n q_n + r} \right)^{N-1}$ if $r < 0$. In any case $|u_n'(r)| \leq c|r|$ then u_n converges (locally uniformly) to a function u that satisfies

$$\begin{cases} -u'' = u, & \text{in } (\alpha_o, +\infty), \\ |u(r)| \leq |u(0)| = 1, \quad u'(0) = 0. \end{cases}$$

This is not possible because $u(r) = \cos r$, which has an infinite number of nodal zones. Eventually we have proved that $\|v_p\|_\infty^{p-1}$ is bounded.

Now we are in position to show that (3.6) and (3.7) hold.

Let p_n be a sequence such that $p_n \rightarrow 1$ as $n \rightarrow +\infty$ and let $v_n := v_{p_n}$. The function $\bar{v}_n = \frac{v_n}{\|v_n\|_\infty}$ satisfies

$$\begin{cases} -\Delta \bar{v}_n = \|v_n\|_\infty^{p_n-1} |\bar{v}_n|^{p_n-1} \bar{v}_n & \text{in } A \\ \bar{v}_n = 0 & \text{on } \partial A. \end{cases}$$

This implies that $\|v_n\|_\infty^{p_n-1}$ can not go to zero otherwise \bar{v}_n would converge uniformly to zero and this is a contradiction with $\|\bar{v}_n\|_\infty = 1$.

Therefore, up to a subsequence, $\|v_n\|_\infty^{p_n-1} \rightarrow \lambda$ and \bar{v}_n converges uniformly in A to a function \bar{v} . Let us show that

$$(3.10) \quad (|\bar{v}_n|^{p_n-1} - 1) \bar{v}_n \rightarrow 0.$$

For any fixed n , we have $(|\bar{v}_n|^{p_n-1} - 1) \bar{v}_n = 0$ if $\bar{v}_n = 0$, otherwise

$$\begin{aligned} |(|\bar{v}_n|^{p_n-1} - 1) \bar{v}_n| &\leq (p_n - 1) \left| \log |\bar{v}_n| \int_0^1 |\bar{v}_n|^{1+t(p_n-1)} dt \right| \\ &\leq c(p_n - 1) |\bar{v}_n|^{1/2} \leq c(p_n - 1). \end{aligned}$$

So obviously $\|v_n\|_\infty^{p_n-1} |\bar{v}_n|^{p_n-1} \bar{v}_n \rightarrow \lambda \bar{v}$ and \bar{v} solves

$$\begin{cases} -\Delta \bar{v} = \lambda \bar{v} & \text{in } A \\ \bar{v} = 0 & \text{on } \partial A. \end{cases}$$

Finally the limit eigenfunction \bar{v} is radial and has exactly m nodal zones. Actually by (3.8) it follows that the last internal zero r_n satisfies $b - r_n > c \|v_n\|_\infty^{\frac{p_n-1}{2}}$ and therefore the last nodal zone can not collapse to a null set. Similarly, this can not happen for all the nodal zones. Inside each zone, \bar{v}_n is strictly positive (or negative) and converges uniformly to \bar{v} . Hence \bar{v} cannot change sign and Hopf Lemma guarantees that no further zero can appear. \square

Next we deduce some information about the asymptotic of the eigenvalues $\nu_l(p)$ as $p \rightarrow 1$.

Lemma 3.3. *For p near to 1 we have that*

$$(3.11) \quad \nu_l(p) \text{ are bounded from below for any } l \geq 1,$$

$$(3.12) \quad \lim_{p \rightarrow 1^+} \nu_m(p) = 0.$$

Proof. To check (3.11) it is enough to show that $\nu_1(p)$ is bounded from below as $p \rightarrow 1$. By definition

$$\nu_1(p) = \inf_{\phi \in H_{0,r}^1(A)} \frac{\int_a^b r^{N-1} (|\phi'|^2 - p|v_p|^{p-1} \phi^2) dr}{\int_a^b r^{N-3} \phi^2 dr}.$$

From Proposition 3.2 we have

$$p|v_p|^{p-1} = p\|v_p\|_\infty^{p-1} |\bar{v}_p|^{p-1} \leq C$$

as $p \rightarrow 1$. Then, for any $\phi \in H_{0,r}^1(A)$ we have as $p \rightarrow 1$

$$\begin{aligned} \int_a^b r^{N-1} (|\phi'|^2 - p|v_p|^{p-1} \phi^2) dr &\geq \int_a^b r^{N-1} (|\phi'|^2 - C\phi^2) dr \\ &\geq -C \int_a^b r^{N-1} \phi^2 dr \geq -cb^2 \int_a^b r^{N-3} \phi^2 dr, \end{aligned}$$

so that

$$\frac{\int_a^b r^{N-1} (|\phi'|^2 - p|v_p|^{p-1} \phi^2) dr}{\int_a^b r^{N-3} \phi^2 dr} \geq -cb^2$$

which gives that $\nu_1(p) \geq -cb^2$.

Next, since we already know that $\nu_m(p) < 0$ for all p , it suffices to check that $\underline{\nu} = \liminf_{p \rightarrow 1^+} \nu_m(p) = 0$. To this end, let $p_n \rightarrow 1$ so that $\nu_n := \nu_m(p_n) \rightarrow \underline{\nu}$ and

let ϕ_n be the m -eigenfunction for problem (3.3) with $p = p_n$, normalized so that $\|\phi_n\|_\infty = 1$. We compare the eigenfunction ϕ_n with $\bar{v}_n = v_{p_n}/\|v_{p_n}\|_\infty$, that satisfies

$$\begin{cases} -\bar{v}_n'' - \frac{N-1}{r}\bar{v}_n' = |v_{p_n}|^{p_n-1}\bar{v}_n & a < r < b, \\ \bar{v}_n(a) = \bar{v}_n(b) = 0. \end{cases}$$

Assume by contradiction that $\underline{\nu} < 0$. Remembering that $\|v_{p_n}\|_\infty^{p_n-1}$ is bounded by Proposition 3.2 we get

$$p_n|v_{p_n}|^{p_n-1} + \frac{\nu_n}{r^2} - |v_{p_n}|^{p_n-1} \leq (p_n - 1)c + \frac{\nu_n}{b^2} \leq 0$$

for n large enough. On the other hand, ϕ_n and \bar{v}_n have the same number of zeros, hence Sturm-Picone comparison theorem yields that $\phi_n = \pm\bar{v}_n$. In particular ϕ_n and \bar{v}_n solve the same equation, that is

$$\nu_n = (p_n - 1)r^2|v_{p_n}|^{p_n-1}.$$

Passing to the limit as $n \rightarrow +\infty$, and using again the boundedness of $|v_{p_n}|^{p_n-1}$, we end up with the contradiction $\underline{\nu} = 0$. \square

Lemma 3.4. *The map $p \mapsto \nu_l(p)$ is locally analytic and the set $\{\nu_l(p) = -j(N-2+j)\}$ consists of only isolated points.*

Proof. By Lemmas 3.1 and 3.3, we have that, for any fixed integer $j \geq 1$, if p solves (1.3) or (1.4) then p belongs to a compact set in $(1, +\infty)$. Then, arguing as in [DW, Lemma 3.3 part (c)], the claim follows. \square

Eventually, putting together the characterization of the degenerate p obtained in Proposition 1.2 with the information collected in Lemmas 3.1, 3.3 and 3.4 we are able to conclude the proof of Proposition 1.3.

Proof of Proposition 1.3. The values of p such that v_p is degenerate are given by the solution of the equations (1.3) or (1.4). By Lemmas 3.1 and 3.3 the equation $\nu_m(p) = -j(N-2+j)$ admits at least a solution for any $j \geq 1$. So the values of p such that v_p is degenerate build up an infinite set, which consists of isolated points by Lemma 3.4.

In addition the Morse index of v_p is given by the formula (3.5), and from Lemma 3.1 it is easy to see that $J_l(p)$ goes to infinity together with p . Then $m(v_p) \rightarrow +\infty$ as $p \rightarrow +\infty$. \square

4. EXISTENCE OF SOLUTIONS IN ANNULAR TYPE DOMAINS

Here we prove our perturbation theorem.

Proof of Theorem 1.1. We start by introducing a change of variable which puts into relation problem (1.1) in Ω_t with a problem in an annulus. For $A = \{x \in \mathbb{R}^N : a < |x| < b\}$ let us consider a smooth function $\sigma : \bar{A} \rightarrow \mathbb{R}^N$ and define

$$(4.1) \quad \Omega_t = \{x + t\sigma(x) : x \in A\}$$

Note that for t small enough Ω_t is diffeomorphic to the annulus A . Moreover there is another smooth function $\tilde{\sigma}$ such that $x = y + t\tilde{\sigma}(y) \in A$ for $y \in \Omega_t$ (at least for small values of t). An immediate computation shows that finding a solution $u(y)$ of (1.1) in Ω_t is equivalent to find a solution v of

$$(4.2) \quad \begin{cases} -\Delta v - L_t v = |v|^{p-1}v & \text{in } A, \\ v = 0 & \text{on } \partial A, \end{cases}$$

where L_t is a linear operator

$$L_tv = t \sum_{i,k} \partial_{y_i y_i}^2 \tilde{\sigma}_k \partial_{x_k} v + 2t \sum_{i,k} \partial_{y_i} \tilde{\sigma}_k \partial_{x_i x_k}^2 v + t^2 \sum_{i,j,k} \partial_{y_j} \tilde{\sigma}_i \partial_{y_j} \tilde{\sigma}_k \partial_{x_i x_k}^2 v.$$

Note that for $t = 0$ problem (4.2) gives back problem (2.1) on the annulus (or (2.2) for $m = 1$). Next we follow [Cow] and define a function $F : \mathbb{R} \times C_0^{2,\gamma}(\bar{A}) \rightarrow C_0^{0,\gamma}(\bar{A})$ by

$$F(t, v) = -\Delta v - L_tv - |v|^{p-1}v.$$

It is easily seen that F is a C^1 map verifying $F(0, v_p) = 0$. In order to apply the Implicit Function Theorem we examine $D_v F(0, v_p)$, the Fréchet derivative of F with respect to $v \in C_0^{2,\gamma}(\bar{A})$ computed at $(0, v_p)$. Its kernel is described by the solutions $w \in C_0^{2,\gamma}(\bar{A})$ to the linearized problem

$$-\Delta w = p |v_p|^{p-1} w.$$

Hence the map $D_v F(0, v_p)$ has a bounded inverse for all p such that the related radial solution v_p is nondegenerate in $C_0^{2,\gamma}(\bar{A})$. For $m > 1$, it follows by Lemmas 1.2 and 3.4 that there is only an increasing sequence of isolated values of p where v_p is degenerate. In the case $m = 1$, the same property was already established in [GGPS].

So the Implicit Function Theorem applies and there is a continuum of functions $v_t \in C_0^{2,\gamma}(\bar{A})$ such that $F(t, v_t) = 0$ and the number of its nodal zones coincides with the ones of v_0 , at least for $|t|$ small, because the map $t \mapsto v_t$ is continuous on $C_0^{2,\gamma}(\bar{A})$. Eventually $u_t(y) := v_t(x)$ is a solution of (1.1) in Ω_t with exactly m nodal zones. \square

We end this section by proving some additional properties of the solution in the perturbed domain.

Proposition 4.1. *Let us consider the solution u_p of problem (1.1) in Ω_t given by Theorem 1.1. Then the Morse index of the solution u_p satisfies*

$$(4.3) \quad m(u_p) = m(v_p)$$

where v_p is the radial solution in the annulus. Finally,

$$(4.4) \quad \lim_{p \rightarrow +\infty} m(u_p) = +\infty$$

Proof. Using the map σ we get that the Morse index of u_p in Ω_t is the same of the corresponding function $v_{t,p}$ in A . Let us show that, for t small, we have that

$$(4.5) \quad m(v_{t,p}) = m(v_p)$$

By contradiction suppose that we have that there exists a sequence $t_n \rightarrow 0$ such that $m(v_{t_n,p}) \neq m(v_p)$. Then, since the Morse index is an integer, we deduce that $\lim_{n \rightarrow +\infty} m(v_{t_n,p}) \neq m(v_p)$. On the other hand, since $v_{t_n,p} \rightarrow v_p$ in $C^2(A)$ as $n \rightarrow +\infty$ we get a contradiction.

Finally (4.4) follows by Lemma 3.1. \square

Our last result provides some information on the shape of the solution in the perturbed annulus Ω_t at least as p is close to 1 and $+\infty$.

Proposition 4.2. *Let u_p be the solution in Ω_t given by Theorem 1.1. Then, for any $\epsilon > 0$*

i) there exist $p_0 = p_0(\epsilon)$ and $t_0 = t_0(\epsilon)$ such that for any $1 < p < p_0$ and $|t| < t_0$ we have

$$(4.6) \quad \|u_p - \psi_m(y + t\tilde{\sigma}(y))\|_{C^0(\Omega_t)} < \epsilon$$

where ψ_m is the function appearing in Proposition 3.2,

ii) there exist $p_0 = p_0(\epsilon)$ and $t_0 = t_0(\epsilon)$ such that for any $p > p_0$ and $|t| < t_0$ we have

$$(4.7) \quad \|u_p - \omega(y + t\tilde{\sigma}(y))\|_{C^0(\Omega_t)} < \epsilon$$

where $\omega(x)$ is the radial function which appears in Theorem 1.1 in [PS].

Proof. We have that (4.6) follows by Proposition 3.2 and Theorem 1.1 and (4.7) follows again by Theorem 1.1 and by the result in [PS]. \square

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